A Stochastic Analysis of an Anharmonic Sensor Phase Response

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***Abstract*—The probability density function (pdf) of a modulo 2π phase response slope of an intrinsic anharmonic sensor of a crystal oscillator is studied in detail. It is noted that without an external drive, the sensor is excited by the oscillator noise floor with a signal-to-noise ratio (SNR) of around unity. The slope pdf is provided both in the rigorous integral form and in the T-distribution-based approximation. It is shown that the slope mean value is estimated to be zero with SNR $= 0$. It then gradually tends toward actual value as SNR rises so that with SNR > 2 the bias of slope estimates is almost negligible. With $0 \leq$ SNR < 0.7, the slope variance stays at a maximum and then asymptotically diminishes toward zero as the SNR rises. The importance of these studies resides in a shown fact that, practically, having SNR < 2 in anharmonic sensors may result in substantial bias and variance for phase response slope mod 2π estimates.**

***Index Terms*—Piezoelectric sensor, stochastic analysis.**

I. Introduction

QUARTZ crystal sensors have been a subject for peer examination in precision electronics for several decades [1]. Owing to great sensitivity, precision, and accuracy, such sensors have gained currency in precise measurements of temperature, pressure, humidity, acceleration, etc. Among all applications, there is a relatively young and not yet well studied field. Here, intrinsic anharmonic resonances of a crystal resonator are considered to be sensors of environment for an oscillator operating at the fundamental mode of the same resonator [2]. Such an idea has been widely employed since the 1970s and, consequently, a lot of solutions were patented. The authors have been concerned primarily with simultaneous excitation of the fundamental and anharmonic mode within the dual-mode [3] or multi-mode oscillator loop. Basically, such a loop works stably with close drive levels in each vibration. It then turned out that this operation principle imposes an important limitation to the approach: having a close drive level, a sensor interacts with the fundamental mode via the resonator thermal field [4] affecting oscillator accuracy. An alternative approach is known as the modulational method [5]. It assumes that a sensor initially operates being excited only by the oscillator noise floor[1]. To increase the signal-to-noise ratio (SNR), a modulating signal is induced in the oscillator loop, having a frequency such as that between the fundamental and sensor modes. The drive level for a sensor is then obtained to be near optimal in a sense of small tracking error and negligible interaction with the fundamental mode. The approach was practically implemented in the quartz crystal standards for adjusting oven temperature via the "B"-mode of the SC-cut resonator [6] and to compensate the fundamental mode aging rate via the anharmonic mode of the AT-cut resonator [7].

In this paper, we address a stochastic investigation of the anharmonic sensor excited by the oscillator noise floor, assuming a noise to be Gaussian. We primarily focus attention on the probabilistic analysis of the sensor phase response slope, as this performance is responsible for the sensor resolution and precision. The paper is organized as follows. In Section II, we consider the signal models, specify the slope of a sensor phase response, and formulate the problem. The slope pdf in presence of Gaussian noise is derived in Section III, in which we give two relevant integral relations. Limiting cases of SNR are also considered here. A useful approximation based on the Tikhonov distribution (T-distribution) can be found in Section IV, along with the asymptotic pdf, providing calculations with a reasonable accuracy via the modified Bessel functions of the first kind zero order. Finally, conclusions are drawn in Section V. Hereby, we answer the major question: which SNR should be enough being provided by modulation to achieve a negligible bias and small noise in the estimate of the modulo 2π sensor phase response slope?

II. Signal Model and Problem Formulation

An anharmonic sensor responds both in the oscillator amplitude and phase. It is, however, much easier in practice to detect and track the sensor via the amplitude [5]. Then, consider an oscillator amplitude spectrum envelope (Fig. 1) and note that with the noise floor drive level, the sensor produces a nonuniformity around the Fourier frequency $f_s = f_s - f_{os}$, where f_s and f_{os} is a sensor resonance frequency and an oscillator frequency, respectively. Fig. 1 shows, as well, the relevant sensor phase response, which is readily measured as the phase angle between the modulating signal [5] and the envelope of an oscillator signal [2].

A sensor response in the oscillator output may be described at an arbitrary frequency $\omega_0 = 2\pi f_0$ around $\omega_s = 2\pi f_s$ as a deterministic harmonic signal

$$s(t) = A(\omega_0)\cos[\omega_0 t + \phi(\omega_0) + \vartheta] \quad (1)$$

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[1]It is tacitly assumed that the resonator fundamental vibrations are excited with normal drive levels and, thus, the physical courses of the specific crystal resonator phenomenon associated with small SNR (low drive levels) [16]–[20] are removed for the anharmonic sensors by the intensively vibrating piezoelectric plate.

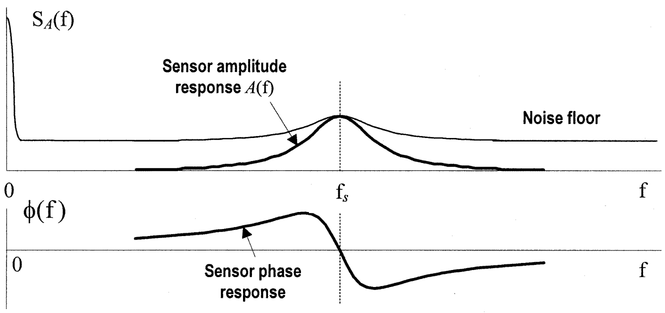

Fig. 1. Anharmonic sensor response in the oscillator spectrum.

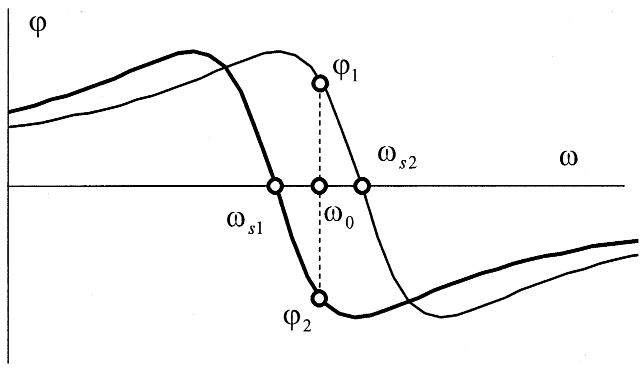

Fig. 2. Phase response of the anharmonic sensor at two different time instants.

where $A(\omega_0)$ and $\phi(\omega_0)$ are, respectively, values of the amplitude and phase responses at ω_0, and ϑ is an initial constant phase. Considering the oscillator noise floor to be Gaussian with known variance σ_v, one may model the noise at as a narrowband Gaussian process

$$v(t) = A(t)\cos[\omega_0 t + \theta(t)] \tag{2}$$

where $0 \le A(t)$ is a random amplitude distributed with the Rayleigh law and $-\pi \le \theta(t) \le \pi$ is uniformly distributed the modulo 2π random phase [8]. Since, practically, a noise (2) perturbs a signal (1) linearly, we write an oscillator output in the form of a narrowband random signal

$$\xi(t) = s(t) + v(t) = V(t,\omega_0)\cos[\omega_0 t + \varphi(t,\omega_0)] \tag{3}$$

in which $V(t,\omega_0)$ is the positive valued envelope, $0 \le V(t,\omega_0)$ and $\varphi(t,\omega_0)$ is the modulo 2π total phase, $-\pi \le \varphi(t,\omega_0) \le \pi$.

To estimate the slope of a phase response $\varphi(t)$ in presence of Gaussian noise, we would like the sensor to be examined at two different time instants. Subsequently, we assume that a sensor responds to the measured parameter (temperature, for example) at the resonance frequency ω_{s1} and ω_{s2} at time t_1 and t_2, respectively, having random phases φ_1 and φ_2 at the intermediate frequency (Fig. 2). This, in turn, allows us to specify the slope by

$$\varphi'_\omega = \lim_{\Delta\omega\to 0}\frac{\varphi_2-\varphi_1}{\omega_{s2}-\omega_{s1}} = \lim_{\Delta\omega\to 0}\frac{\Delta\varphi}{\Delta\omega} \tag{4}$$

where $\Delta\omega = \omega_{s2}-\omega_{s1}$, $\Delta\varphi = \varphi_2 - \varphi_1$ is a difference phase mod 2π and φ_1 and φ_2 are supposed to be statistically independent processes mod 2π being taken at different times[2]. Either phase φ_1 or φ_2 mod 2π inherently lies between $-\pi$ to π. Consequently, their difference $\Delta\varphi$ ranges from -2π to 2π [9]. In the measurement set, each phase angle is measured separately and, thus, the actual distribution of $\Delta\varphi$ may be calculated. In the phase (sensor) tracking system, however, $\Delta\varphi$ is estimated to lie from $-\pi$ to π [8], except for in special cases. We will then be interested throughout the paper only by the $\Delta\varphi$ mod. 2π

III. PDF OF THE PHASE RESPONSE SLOPE PERTURBED BY GAUSSIAN NOISE

Let us now derive the pdf of the slope (4) and estimate its major statistics, mean, and variance. A slope is normally calculated for the finite frequency span $\Delta\omega/2$ taken to the left and to the right of the resonance frequency in the linear range of the phase response. This means that without a loss of generality, one may consider $\varphi'_\omega = \Delta\varphi/\Delta\omega$ and, thus, (4) degenerates to the reduced slope

$$z = \Delta\varphi = \varphi'_\omega \Delta\omega. \tag{5}$$

Furthermore, since each frequency ω_{s1} and ω_{s2} is equidistant to ω_0 (Fig. 2) and shape of $A(\omega)$ may be assumed to be symmetric (Fig. 1), we may consider a SNR[3] $a = A(\omega)/\sqrt{2}\sigma_v$ to be identical in either phase, this is $A_1(\omega_0)/\sigma_v\sqrt{2} = A_2(\omega_0)/\sigma_v\sqrt{2} = a_1$.

Relation (5) claims that, as $\Delta\omega$ is constant, distributions of $\varphi'_\omega(t)$ and $\Delta\varphi(t)$ are shape equal, and the task then requires a probabilistic analysis of the phase shift between two harmonic signals of the same frequency perturbed by Gaussian noise. Such a problem coincides with those in analog communications (problems in angle modulation and demodulation), digital communications (bit error probability [9]), pulse radars, phase-locked loops (PLL) [8], and some other areas. In the overview work [9], Pawula *et al* have provided an extensive examination of the possible solutions for arbitrary parameters of signals (1) and (2). An analysis was done starting from the rigorous integral pdf of $\Delta\varphi$ derived by Fleck and Trabka in 1961 [10] and the fundamental Bennett pdf of phase [11]. Some new useful results were then published by Pawula in [12].

In our particular case of φ_1 and φ_2 taken at different times, we should assume them to be independent, and then their joint pdf to be a multiplication of one-dimensional pdf

$$p(\varphi_1,\varphi_2) = p(\varphi_1)p(\varphi_2). \tag{6}$$

Equation (6) turns us back to the early Bennett work [11], in which he has shown that the phase pdf is subject to the analytic law, having the form, for example, of [8]

$$p(\varphi) = \frac{e^{-a^2}}{2\pi}\left[1 + 2a\sqrt{\pi}e^{a^2\cos^2\varphi}\Phi(a\sqrt{2}\cos\varphi)\cos\varphi\right] \tag{7}$$

in which $\Phi(x)$ is a probability integral (A1.1).

Taking (7) into account, in order to determine pdf of φ'_ω, let us transfer from the system of functions φ_1 and φ_2 to that of

[2]This holds true for the assumed short correlation time Gauss noise.

[3]Throughout the paper, we use this definition of SNR [8] whereas in communications [9] it is specified for the signal and noise power to be a^2.

φ_1 and φ'_ω. From (5), we get $\varphi_2 = \varphi_1 + \Delta\omega\varphi'_\omega$, and then the determinant of the Jacobian of the transformations becomes $J = |\partial(\varphi_1,\varphi_2)/\partial(\varphi_1,\varphi'_\omega)| = \begin{vmatrix} 1 & 0 \\ 1 & \Delta\omega \end{vmatrix} = \Delta\omega$, transferring (6) to

$$p(\varphi_1, \varphi'_\omega) = \Delta\omega p(\varphi_1) p(\varphi_1 + \varphi'_\omega \Delta\omega). \tag{8}$$

Now, we use (5), designate $\varphi \equiv \varphi_1$ and $\varphi_2 = \varphi + z$, and derive the required one-dimensional (1-D) pdf of the reduced slope by integrating (8) over φ from $-\pi$ to $+\pi$

$$p(\varphi'_\omega) = \Delta\omega \int_{-\pi}^{\pi} p(\varphi) p(\varphi + \varphi'_\omega \Delta\omega) d\varphi. \tag{9}$$

Involving (5) then leads to the formula given in [9, (6)] for $\Delta\varphi$, that is

$$p(z) = \int_{-\pi}^{\pi} p(\varphi) p(\varphi + z) d\varphi \tag{10}$$

in which pdf $p(\varphi)$ and $p(\varphi + z)$ is readily derived from (7) to be

$$p(\varphi) = A_1 \left[1 + 2a_1\sqrt{\pi} x_1(\varphi)\right] \tag{11}$$
$$p(\varphi + z) = A_1 \left[1 + 2a_1\sqrt{\pi} x_2(\varphi + z)\right] \tag{12}$$

where $A_1 = e^{-a_1^2}/2\pi$ and the auxiliary probability functions are

$$x_1(\varphi) = e^{a_1^2 \cos^2(\varphi - \phi_1)} \Phi\left[a_1\sqrt{2}\cos(\varphi - \phi_1)\right] \times \cos(\varphi - \phi_1), \tag{13}$$
$$x_2(\varphi + z) = e^{a_1^2 \cos^2(\varphi - \phi_2 + z)} \Phi\left[a_1\sqrt{2}\cos(\varphi - \phi_2 + z)\right] \times \cos(\varphi - \phi_2 + z). \tag{14}$$

Substituting (11) and (12) into (10) yields

$$p(z) = \frac{1}{2\pi}\left[e^{-a_1^2}\left(2 - e^{-a_1^2}\right) + 2a_1^2 e^{-2a_1^2} \int_{-\pi}^{\pi} x_1(\varphi) x_2(\varphi + z) d\varphi\right] \tag{15a}$$

and we may note for a fact that the integral in (15a) cannot be solved in simple functions because of the integrand combined with the shifted probability integrals multiplied by the exponential and harmonic functions. It may only be reduced to a slightly simpler integrand such as that shown, for example, by Fleck and Trabka in [10][4]

$$p(z) = \frac{e^{-2a_1^2}}{4\pi} \int_0^{\pi} \left[1 + 2a_1^2 + 2a_1^2 \cos(z - \phi_2 + \phi_1)\sin\varphi\right] \times e^{2a_1^2 \cos(z - \phi_2 + \phi_1)\sin\varphi} \sin\varphi d\varphi. \tag{15b}$$

Based upon this fact, we give below an analysis of two limiting cases.

[4]The first rigorous integral pdf was derived by Tsvetnov [20] and others are given in [9], [12], and [13].

A. Asymptotics

Assuming great noise (small SNR), $a_1 \ll 1$, the integral in (15a) tends toward zero and pdf becomes uniform

$$p(z)\big|_{a_1 \ll 1} \cong \frac{1}{2\pi} e^{-a_1^2}\left(2 - e^{-a_1^2}\right)\Big|_{a_1=0} = \frac{1}{2\pi}. \tag{16}$$

Otherwise, with small noise (large SNR), $1 \ll a_1$, the first term in brackets of (15a) tends toward zero whereas the remainder for $\phi \equiv \phi_1$ and $\phi_2 = \phi + \Delta\phi$ transforms to

$$p(z)\big|_{a_1 \gg 1} \cong \frac{a_1^2}{\pi} e^{-2a_1^2} \int_{-\pi}^{\pi} e^{a_1^2[\cos^2(\varphi-\phi) + \cos^2(\varphi - \phi + z - \Delta\phi)]} \times \Phi\left[a_1 \cos(\varphi - \phi)\right] \times \Phi\left[a_1\cos(\varphi - \phi + z - \Delta\phi)\right] \times \cos(\varphi - \phi)\cos(\varphi - \phi + z - \Delta\phi)\, d\varphi. \tag{17}$$

With great SNR, the process may be assumed to be noiseless. Instantly, both probability integrals in (17) become unity. Yet, by $\varphi - \phi \ll 1$ and $\varphi - \phi + z - \Delta\phi \ll 1$, a multiplication of the cosine functions tends toward unity, as well. Now use the decompositions $\cos^2(\varphi - \phi) \cong 1 - (\varphi - \phi)^2$ and $\cos^2(\varphi - \phi + z - \Delta\phi) \cong 1 - (\varphi - \phi + z - \Delta\phi)^2$, and rewrite (17) as

$$p(z)\big|_{a_1 \gg 1} = \frac{a_1^2}{\pi} \int_{-\pi}^{\pi} e^{-a_1^2[2(\varphi-\phi)^2 + 2(\varphi-\phi)(z-\Delta\phi) + (z - \Delta\phi)^2]} d\phi. \tag{18}$$

Integral (A1.2) then brings (18) to the form of

$$p(z)\big|_{a_1 \gg 1} \cong \frac{a_1}{2\sqrt{2\pi}} e^{-a_1^2(z-\Delta\phi)^2/2} \times \left\{\operatorname{erf}\left[\sqrt{2}a_1(\pi - \phi) + \frac{a_1(z - \Delta\phi)}{\sqrt{2}}\right] - \operatorname{erf}\left[-\sqrt{2}a_1(\pi + \phi) + \frac{a_1(z - \Delta\phi}{\sqrt{2}}\right]\right\} \tag{19}$$

in which $\operatorname{erf}(x)$ is a probability function (A1.3). It may easily be shown now that $\operatorname{erf}(x_1) - \operatorname{erf}(x_2) \to 2$, once with $a_1 \to \infty$ the variables of the first and the second erf-functions become $+\infty$ and $-\infty$, respectively. Accordingly, (18) converges to

$$p(z)\big|_{a_1 \gg 1} \cong \frac{a_1}{\sqrt{2\pi}} \exp\left[-\frac{a_1^2(z - \Delta\phi)^2}{2}\right] \tag{20}$$

that is nothing less then the normal density $p_1(z) = 1/\sqrt{2\pi\sigma^2} \exp\left[-(z - \Delta\phi)^2/2\sigma^2\right]$ with an expectation $\Delta\phi$ and variance

$$\sigma^2 = \frac{1}{a_1^2}. \tag{21}$$

It is to be remarked now that a simplified form of (15a) and its asymptotic forms were given by Pawula in [13, (A10), (A11a–c)], in which a large SNR tends (A11a) to (20). We then conclude that as SNR rises, the pdf (15a) normalizes, and its variance decreases.

IV. T-Distribution-Based Approximation

So far, we dealt with the rigorous pdf, in which either asymptotic (16) or (20) is reasonably accurate in the range $a_1 < 0.2$ and $2 < a_1$, respectively. Herewith, the gap $0.2 < a_1 < 2$ calculates accurately only by (15a) or by equal integral relations that was mentioned more than once in literature ([9], [10], [12], [13], and [20]). To achieve at least an engineering approximation, let us recall that the T-distribution by Tikhonov [14] serves for the same problem in PLL phase stochastic theory, that is

$$p^T(\varphi) = \frac{1}{2\pi I_0(D)} \exp\left[D\cos(\varphi - \phi)\right] \quad (22)$$

where $I_0(D)$ is the modified Bessel function of the first kind zero order and D is a parameter dependent on SNR. Approximating each of the terms in the integrand of (10) by (22) yields

$$p^a(z) = \frac{1}{4\pi^2 I_0^2(D)} \int_{-\pi-\phi}^{\pi-\phi} e^{D[\cos x + \cos(x+z-\Delta\phi)]} dx, \quad (23)$$

where $D = b_0 + b_1 a_1 + b_2 a_1^2$; $b_0 \approx 0.1715$, $b_1 \approx 0.1125$, and $b_2 \approx 1.9912$ are coefficients of the approximating polynomial [14]. Now introduce $d_1 = D[1 + \cos(z - \Delta\phi)]$ and $d_2 = D\sin(z - \Delta\phi)$, and first go to the function

$$p^a(z) = \frac{1}{4\pi^2 I_0^2(D)} \int_{-\pi-\phi}^{\pi-\phi} e^{d_1 \cos x - d_2 \sin x} dx. \quad (24)$$

Then substitute $d_1 \cos x - d_2 \sin x = r\cos(x - \vartheta)$, where $r = \sqrt{d_1^2 + d_2^2}$ and $\vartheta = -\mathrm{arctg}(d_2/d_1)$, bring integral in (24) to the Bessel function (A1.4), and finally derive a pdf

$$p^a(z) = \frac{I_0(r)}{2\pi I_0^2(D)} \quad (25)$$

in which $r = D\sqrt{2[1 + \cos(z - \Delta\phi)]}$. On the first glance, this form (25) is superior to that of (15a) or other similar integral relations from the engineering point of view. The only question remains the approximation error. Below, we provide the relevant analysis, starting with the same asymptotic cases as for (15a).

A. Asymptotics of the T-based Approximation

In the first limiting case, $a_1 \ll 1$, we get $D \ll 1$, $I_0(D) \cong 1$, and $I_0(r) \cong 1$. Thus, (25) reaches the same uniform pdf (16). In the second case, $a_1 \gg 1$, first use an approximation (A1.5) and then suppose $I_0(D) \cong e^{2a_1^2}/2a_1\sqrt{\pi}$ and $I_0(r) \cong e^{4a_1^2[1-(z-\Delta\phi)^2/8]}/2a_1\sqrt{2\pi\,[1-(z-\Delta\phi)^2/8]}$. Instantly, (25) transforms to the normal law (20) with the same variance (21). Hence, as well as in the rigorous case (15a), pdf (25) demonstrates the same asymptotics and, so, it is also accurate in the aforementioned SNR ranges $a < 0.2$ and $2 < a$

To illustrate errors in the gap of $0.2 < a > 2$, Fig. 3 shows an assemblage of (15a) and (25) for several SNR with $\Delta\phi = -\pi/2$, for example. It turns out that the error related to the pdf value with $z = \Delta\phi$ appears to be of about 8% in a gap of $0.35 < \mathrm{SNR} < 0.6$, $\approx 6\%$ for $\mathrm{SNR} = 0.7$, $\approx 3\%$ for $\mathrm{SNR} = 2$, and $<2\%$ for $2.8 \leq \mathrm{SNR}$.

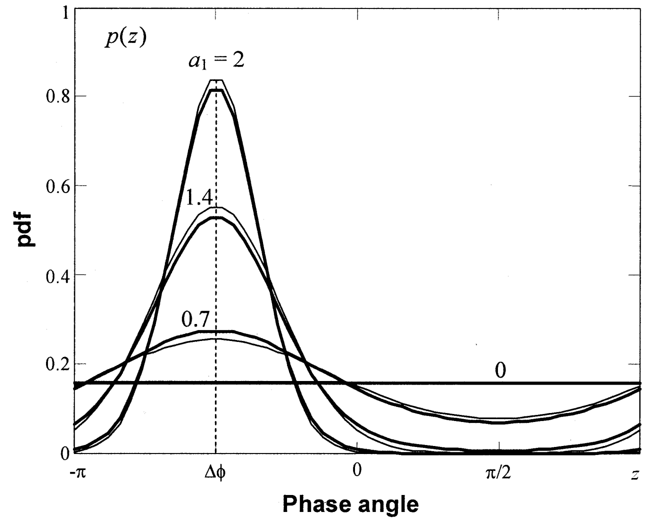

Fig. 3. Pdf of the sensor phase response reduced slope for several SNR: (15a) dark and (25) light .

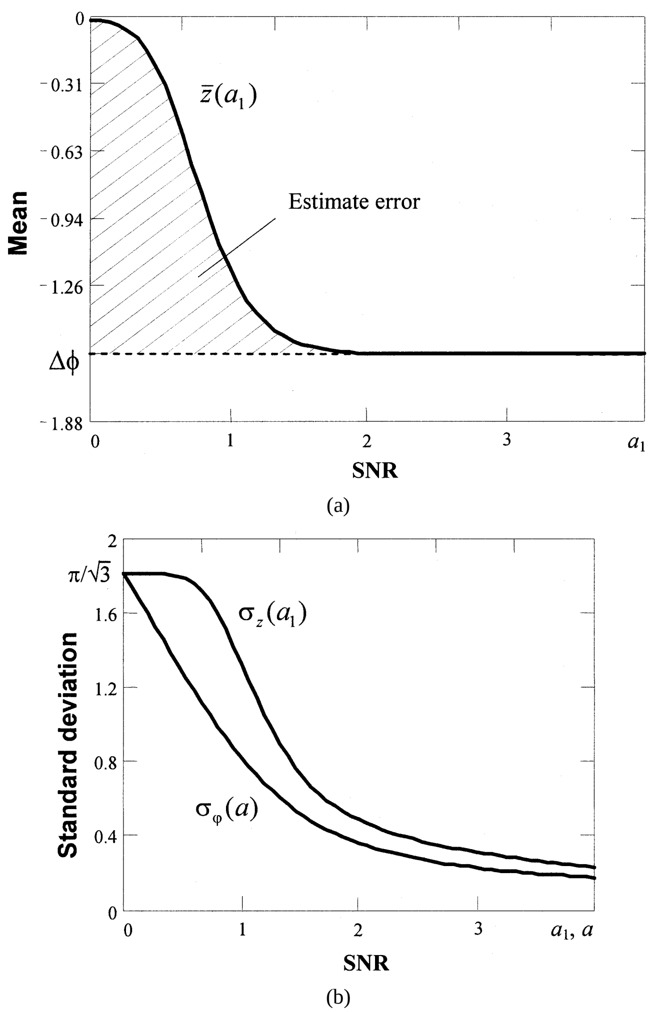

Fig. 4. General statistics. (a) Mean slope $\bar{z}(a_1)$. (b) Standard deviation of the phase and reduced slope, $\sigma_{\varphi(a)}$ and $\sigma_z(a_1)$, respectively.

B. Mean and Variance

Let us now estimate the slope mean and variance. Working out with $\bar{z} = \int_{-\pi}^{\pi} z p_1^a(z) dz$, we come, however, to the infinite series

$$\bar{z}(a_1) = \frac{2}{I_0^2(D)} \sum_{i=1}^{\infty} i^{-1}(-1)^{i+1} I_i^2(D) \sin(i\Delta\phi) \quad (26)$$

and note that variance cannot be expressed in a simple form as well. Numerical calculus in Fig. 4 illustrates the mean and variance in the range of a small SNR, $a_1 \leq 4$. It turns out that the mean slope changes slowly when noise is great $(a_1 < 1)$ and practically reaches $\Delta\phi$ with $2 < a_1$. A simple empirical approximation is available here: $\bar{z}(a_1) \cong \Delta\phi 2.34 a_1^4/(1 + 2.34 a_1^4)$. Contrary to the phase standard deviation $\sigma_\varphi(a)$ given in [8, (7)], the reduced slope standard deviation $\sigma_z(a_1)$ remains almost constant with large noise, $0 \leq a_1 < 0.7$, then decreases rapidly up to $a_1 = 2$ as SNR rises, and finally tends toward zero asymptotically and very slowly when $0.7 \ll a_1$. On the whole, the slope exhibits larger variance as compared to that of the phase itself, excluding the isolated case of $\mathrm{SNR} = 0$, in which the standard deviation in either slope reaches $\pi/\sqrt{3}$.

V. Conclusion

In this paper, we have statistically examined the pdf of the phase response slope of an anharmonic sensor operating in the precision crystal oscillator with SNR of around unity. The rigorous integral pdf was derived and we have shown that the T-distribution may serve as an engineering approximation being performed only by the modified Bessel functions of the first kind zero order. It must be emphasized that having small SNR in the anharmonic crystal sensor, $0 \leq a_1 < 2$, may result in substantial bias and variance for the estimates of the phase response slope owing to its 2π-periodicity. Yet, both estimates appear to be larger than those for the phase response itself. These notations are applicable as well for the conditions of the sensor passive and active tracking. In either case, extremely small SNR leads to almost full insensitivity of the phase mod 2π tracking system [5] [15] and [8], and only with $2 <$ SNR does its tracking performance improve substantially.

Appendix
Mathematical Formulas

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt \quad (A1.1)$$

$$\int e^{-(ax^2+bx+c)} dx = \frac{1}{2}\sqrt{\frac{\pi}{a}} e^{(b^2-4ac)/4a} \times \mathrm{erf}\left(x\sqrt{a} + \frac{b}{2\sqrt{a}}\right) \quad (A1.2)$$

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

$$= 2\Phi\left(x\sqrt{2}\right) - 1 \quad (A1.3)$$

$$I_0(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{x\cos(\varphi-\vartheta)} d\varphi \quad (A1.4)$$

$$I_n(x)\big|_{x\gg 1} \approx \frac{e^x}{\sqrt{2\pi x}} \quad (A1.5)$$

[5]This phenomenon was first mentioned by Tikhonov in [14] for PLL.

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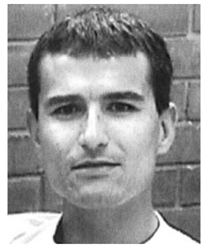

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